A Unified View on the Functional Nerve Theorem and its Variations

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Abstract

The nerve theorem is a basic result of algebraic topology that plays a central role in computational and applied aspects of the subject. In applied topology, one often needs a nerve theorem that is functorial in an appropriate sense, and furthermore one often needs a nerve theorem for closed covers, as well as for open covers. While the techniques for proving such functorial nerve theorems have long been available, there is unfortunately no general-purpose, explicit treatment of this topic in the literature. We address this by proving a variety of functorial nerve theorems. First, we show how one can use relatively elementary techniques to prove nerve theorems for covers by compact, convex sets in Euclidean space, and for covers of a simplicial complex by subcomplexes. Then, we prove a more general, "unified" nerve theorem that recovers both of these, using standard techniques from abstract homotopy theory.

Preliminaries

Definition. Let $\mathcal{A} = (A_i)_{i \in I}$ be a cover of a topological space X. The *nerve* of $\mathcal{A}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{A}) = \{J \subseteq I \mid |J| < \infty,\ A_J := \bigcap_{j \in J} A_j \neq \emptyset\}.$$

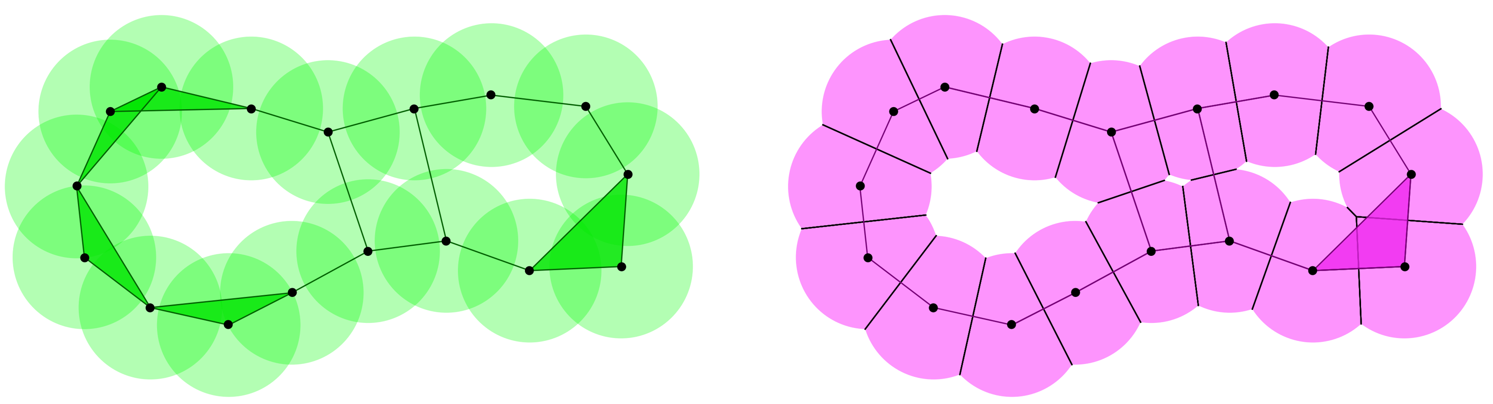


Figure 1: The Čech complex (left) and the Delaunay complex (right).

Definition. For a cover $\mathcal{A} = (A_i)_{i \in [n]}$ of X, the *blowup complex* is

$$\mathrm{Blowup}(\mathcal{A}) = \bigcup_{J \in \mathrm{Nrv}(\mathcal{A})} A_J \times \Delta^{|J|-1} \subseteq X \times \Delta^n.$$

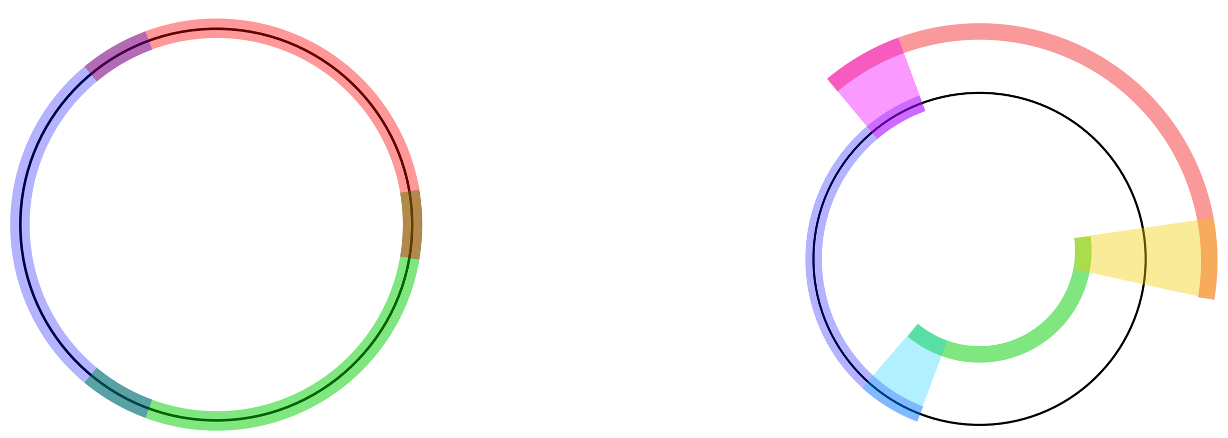


Figure 2: A cover of S^1 (left) and the associated blowup complex (right).

- There are two natural projections:

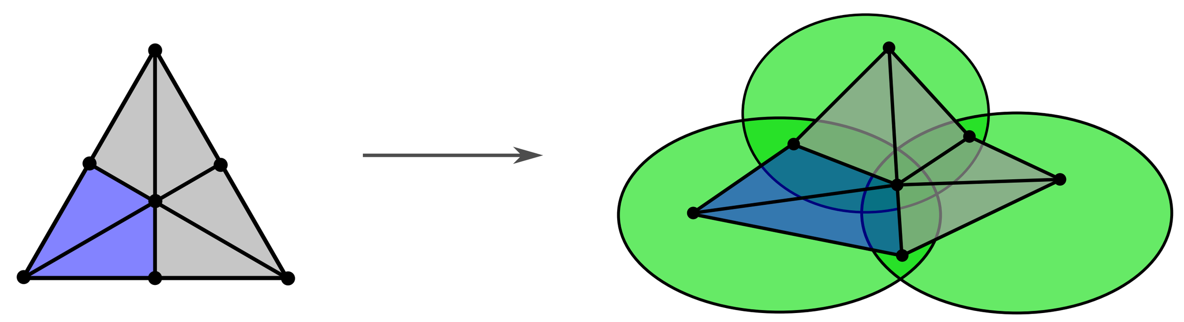
$$X \xleftarrow{\rho_S} \mathrm{Blowup}(\mathcal{A}) \xrightarrow{\rho_N} |\mathrm{Nrv}(\mathcal{A})|$$

- The blowup complex is (not naturally) homeomorphic to the *bar construction* of the nerve diagram → can exploit its good homotopy theoretic properties to prove nerve theorems

A nerve theorem for closed convex sets

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then $|\mathrm{Nrv}(\mathcal{A})|$ is homotopy equivalent to X.

- Define piecewise linear $\Gamma \colon |\mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A})| \to X$ with $\Gamma(\mathrm{bst}\, v_i) \subseteq C_i$.



- Define $\Phi \colon X \to |\mathrm{Nrv}(\mathcal{A})|$ using a partition of unity subordinate to an open thickening of the C_i with $\Phi(C_i) \subseteq \mathrm{bst}\, v_i$.
- Show that Φ is a homotopy inverse to Γ.

Functoriality of nerve theorems

A morphism $(f, \varphi) \colon (X, (A_i)_{i \in I}) \to (Y, (B_j)_{j \in J})$ in the *category of covered spaces* Cov consists of a continuous map $f \colon X \to Y$ and a set map $\varphi \colon I \to J$ such that $f(A_i) \subseteq B_{\varphi(i)}$.

- Space, nerve, and blowup complex are functors Cov → Top.
- There is a zig-zag of natural transformations:

$$\begin{array}{ccccc} X & \longleftarrow & \mathrm{Blowup}(\mathcal{A}) & \longrightarrow & |\mathrm{Nrv}(\mathcal{A})| \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle |\varphi_*|} \\ Y & \longleftarrow & \mathrm{Blowup}(\mathcal{B}) & \longrightarrow & |\mathrm{Nrv}(\mathcal{B})| \end{array}$$

- There are no natural transformations between space and nerve.
- In the special case of closed convex cover filtrations, we prove a one-arrow functorial nerve theorem.

A unified nerve theorem

By using standard techniques from abstract homotopy theory, one can prove many variations of the nerve theorem simultaneously:

Theorem. Let $\mathcal{A} = (A_i)_{i \in I}$ be a cover of the topological space X.

1. Consider the natural map $\rho_S \colon \mathrm{Blowup}(\mathcal{A}) \to X$.
   a) If $\mathcal{A}$ is an open cover, then ρ_S is a weak homotopy equivalence. If furthermore X is paracompact, then ρ_S is a homotopy equivalence.
   b) Let X be compactly generated, and $\mathcal{A}$ a closed cover that is locally finite and locally finite dimensional. If for any $T \in \mathrm{Nrv}(\mathcal{A})$ the latching space $L(T) := \bigcup_{T \subsetneq J \subseteq I} A_J \subseteq A_T$ is closed and the pair $(A_T, L(T))$ satisfies the homotopy extension property, then ρ_S is a homotopy equivalence.
2. Consider the natural map $\rho_N \colon \mathrm{Blowup}(\mathcal{A}) \to |\mathrm{Nrv}(\mathcal{A})|$.
   a) If $\mathcal{A}$ is (weakly) good, then ρ_N is a (weak) homotopy equivalence.
   b) If for all $J \in \mathrm{Nrv}(\mathcal{A})$ the space A_J is compactly generated and $\mathcal{A}$ is homologically good with respect to a coefficient ring R, then ρ_N is an R-homology isomorphism.

Selected references

The literature on the nerve theorem is extensive but unfortunately hard to navigate. For more references and a discussion of the literature on functorial nerve theorems see our paper arXiv:2203.03571.

[1] P. Alexandroff. "Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren geometrischen Anschauung" (1928).

[2] K. Borsuk. "On the imbedding of systems of compacta in simplicial complexes" (1948).