A Unified View on the Functorial Nerve Theorem and its Variations

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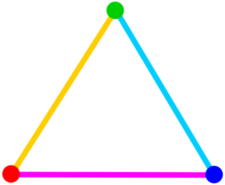
The Alexandrov nerve (1928)

Definition. Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } U_J := \bigcap_{i \in J} U_i \neq \emptyset\}$$

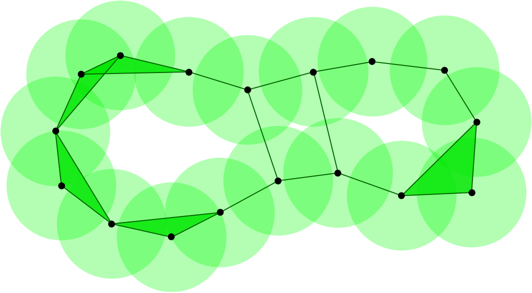


The Alexandrov nerve (1928)

Čech complex of a point cloud

Definition. The *Čech complex* of a subset $X \subseteq \mathbb{R}^d$ is the nerve of the cover by closed balls of radius r centered at points in X

$$\check{\mathrm{C}}\mathrm{ech}_r(X) = \mathrm{Nrv}((D_r(X))_{x \in X}))$$



The history of the nerve theorem

Theorem (Borsuk 1948, and many more). Let $\mathcal{U}$ be a nice cover of X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

Here *nice* can mean different things:

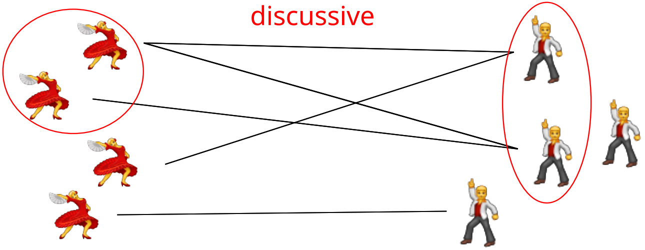
- open, numerable cover, contractible intersections $\rightarrow$ many references
- finite, closed, convex cover $\rightarrow$ few references, mostly using outdated language and tools

Prior results?

- Alexandrov 1928: Every compact metric space is the inverse limit of a sequence of nerves of "arbitrarily fine" closed covers.
- Čech 1932: Extends Alexandrov's ideas $\rightarrow$ Čech (co)homology

Applications to combinatorics

A puzzle from Gavin Wraith



- $W_n = \#\{\text{discussive groups of } n \text{ women}\}$
- $M_n = \#\{\text{discussive groups of } n \text{ men}\}$

Show

$$W_1 - W_2 + W_3 - \cdots = M_1 - M_2 + M_3 - \cdots$$

$$4 - 2 + 0 = 3 - 1 + 0$$

Applications to combinatorics

Definition. Let $S \subseteq A \times B$ be a relation. Define two simplicial complexes

$$L = \{\sigma \subseteq A \mid \exists b \in B : \sigma \times \{b\} \subseteq S\}$$
$$R = \{\tau \subseteq B \mid \exists a \in A : \{a\} \times \tau \subseteq S\}$$

Theorem (Dowker 1951). These are homotopy equivalent $L \simeq R$.

Proof. Consider the good cover $\mathcal{U} = (L_b)_{b \in B}$ of L, where $L_b = \{a \in A \mid (a, b) \in S\}$. By the nerve theorem, $L \simeq \mathrm{Nrv}(\mathcal{U}) \cong R$.

More applications?

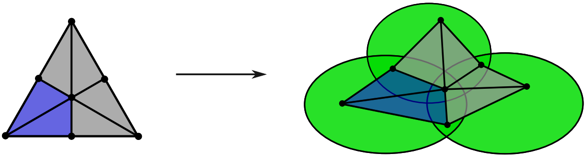
- Nerve theorem based proof of Rota's crosscut theorem
- Nerves appear in Lovász' proof (1978) of Kneser's conjecture (1955)
  $\rightarrow$ emergence of topological combinatorics

Nerve theorem for closed convex sets

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then $\operatorname{Nrv}(\mathcal{A})$ is homotopy equivalent to X.

Proof strategy:

- Construct piecewise linear $\Gamma\colon \operatorname{Sd}\operatorname{Nrv}(\mathcal{A}) \to X$ with $\Gamma(\operatorname{bst} v_i) \subseteq C_i$.



- Construct $\Phi\colon X \to \operatorname{Nrv}(\mathcal{A})$ using a partition of unity subordinate to an open thickening of the C_i with $\Phi(C_i) \subseteq \operatorname{bst} v_i$.
- Show that Φ is a homotopy inverse to Γ.

Nerve theorem for closed convex sets

Some proof details

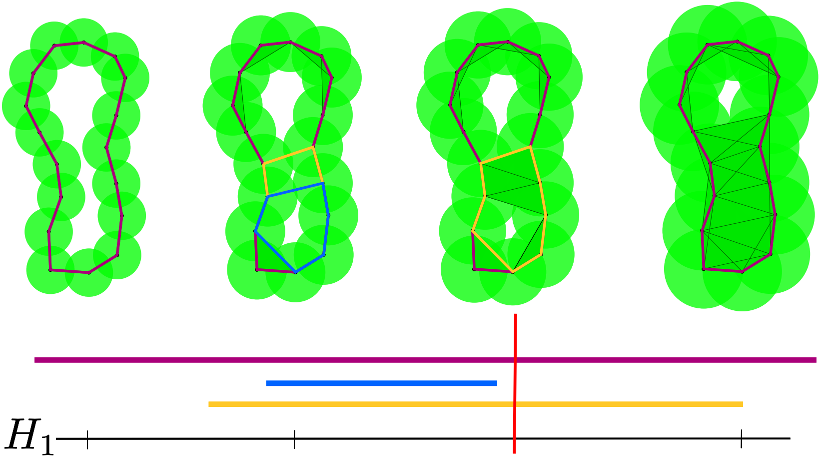
- Take a collection of open sets $\mathcal{G} = (U_i \supseteq C_i)_{i\in[n]}$ satisfying $\operatorname{Nrv}(\mathcal{A}) = \operatorname{Nrv}(\mathcal{G}_\varepsilon)$.
- Take a partition of unity $(\psi_i)_{i\in[n]}$ on X subordinate to the cover $(U_i \cap X)_{i\in[n]}$ of X with $\psi_i|_{C_i} \equiv 1$.
- Define the map $\Phi\colon X \to \operatorname{Nrv}(\mathcal{A})$ by

$$\Phi\colon x \mapsto \sum_{i=0}^{n} \psi_i(x) \cdot v_i$$

- Then $\Gamma \circ \Phi(C_i) \subseteq C_i$ and $\Gamma \circ \Phi \simeq \mathrm{id}_X$ by a straight line homotopy.
- Then $\Phi \circ \Gamma(\operatorname{bst} v_i) \subseteq \operatorname{bst} v_i$ and $\Phi \circ \Gamma \simeq \mathrm{id}_{\operatorname{Sd}\operatorname{Nrv}(\mathcal{A})}$ by induction over the skeleton of $\operatorname{Sd}\operatorname{Nrv}(\mathcal{A})$; use that $(\operatorname{bst} v)_{v\in\operatorname{Vert}\operatorname{Nrv}(\mathcal{A})}$ is a good cover of $\operatorname{Sd}\operatorname{Nrv}(\mathcal{A})$.

Functoriality

Persistent homology



Functoriality

Let $X \subseteq \mathbb{R}^d$, $\mathcal{U}_r = \{D_r(x)\}_{x \in X}$, and $X_r = \bigcup \mathcal{U}_r$. The nerve theorem guarantees for $r \leq l$ the homotopy equivalences

$$\begin{array}{ccc} \mathrm{Nrv}(\mathcal{U}_r) & \longrightarrow & \mathrm{Nrv}(\mathcal{U}_l) \\ \simeq\Big\uparrow & \circlearrowright ? & \Big\uparrow\simeq \\ X_r & \longrightarrow & X_l \end{array}$$

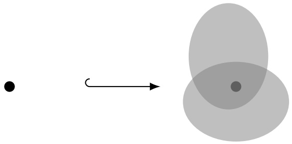
Theorem (Chazal–Oudot 2008). For open coves, this diagram commutes up to homotopy. Hence, it commutes after applying homology.

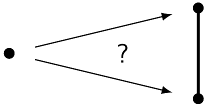
We address two issues:

1. Closed covers were not well-treated in the literature
2. No "proper" functoriality $\rightarrow$ needed in some homotopy-theoretic approaches to TDA (e.g. Blumberg–Lesnick)

Functoriality

Category of covered spaces





Definition. $(U_i)_{i \in I}$ a cover of X, and $(V_\ell)_{\ell \in L}$ a cover of Y. A *map of indexed covers* $\varphi\colon (U_i)_{i \in I} \to (V_\ell)_{\ell \in L}$ is formally a map $\varphi\colon I \to L$. A continuous map $f\colon X \to Y$ is *carried by* φ if $f(U_i) \subseteq V_{\varphi(i)}$.

Definition. The *category of covered spaces* Cov has

- Obj: pairs of the form $(X, (U_i))$, with (U_i) a cover of X
- Mor: $(f, \varphi)\colon (X, (U_i)) \to (Y, (V_\ell))$, continuous map $f\colon X \to Y$ carried by $\varphi\colon (U_i) \to (V_\ell)$

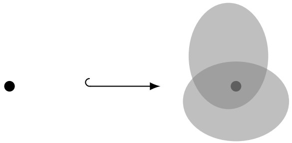
Functoriality

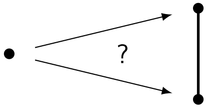
Category of covered spaces

Two functors

- Forgetting the cover: $\mathrm{Spc}\colon$ Cov $\to$ Top, $(X, \mathcal{U}) \mapsto X$
- The nerve: $\mathrm{Nrv}\colon$ Cov $\to$ Top, $(X, \mathcal{U}) \mapsto \mathrm{Nrv}(\mathcal{U})$

Remark. There are no natural transformations $\mathrm{Spc} \Rightarrow \mathrm{Nrv}$

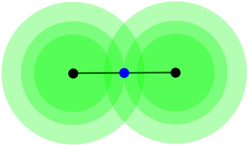




and similarly no natural transformations $\mathrm{Nrv} \Rightarrow \mathrm{Spc}$.

Functoriality

Pointed covers



Definition. Only consider $X \subseteq \mathbb{R}^d$. The category ClConv$_\bullet$ has

- Obj: $(X, \mathcal{A}_\bullet)$, $\mathcal{A}_\bullet$ a finite closed, convex, and *pointed cover* of X
- Mor: $(f, \varphi)\colon (X, \mathcal{A}_\bullet) \to (Y, \mathcal{B}_\bullet)$, $f\colon X \to Y$ carried by $\varphi\colon \mathcal{A} \to \mathcal{B}$:
  - f preserves the basepoints
  - f is affine linear on each cover element

The map $\Gamma\colon \operatorname{Sd}\operatorname{Nrv}(\mathcal{A}) \xrightarrow{\simeq} X$ is natural w.r.t morphisms in ClConv$_\bullet$.

Theorem. On ClConv$_\bullet$ there exists a pointwise homotopy equivalence

$$\operatorname{Sd}\operatorname{Nrv} \Rightarrow \operatorname{Spc}$$

Functoriality

Blowup complex

Definition. For a finite cover $\mathcal{U} = (U_i)_{i \in [n]}$ of X, the *blowup complex* is

$$\mathrm{Blowup}(\mathcal{U}) = \bigcup_{J \in \mathrm{Nrv}(\mathcal{U})} U_J \times \Delta^{|J|-1} \subseteq X \times \Delta^n ,$$

yielding a functor $\mathrm{Blowup}\colon \mathsf{Cov} \to \mathsf{Top}$.

Remark. The blowup complex is (not naturally) homeomorphic to the *bar construction* of the nerve diagram $\to$ can exploit its good homotopy theoretic properties to prove nerve theorems

Functoriality

Any $(f, \varphi)\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ induces a commuting diagram

$$\begin{array}{ccccc} X & \xleftarrow{\rho_S} & \operatorname{Blowup}(\mathcal{U}) & \xrightarrow{\rho_N} & \operatorname{Nrv}(\mathcal{U}) \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi_*} \\ Y & \xleftarrow[\rho_S]{} & \operatorname{Blowup}(\mathcal{V}) & \xrightarrow[\rho_N]{} & \operatorname{Nrv}(\mathcal{V}) \end{array}$$

Hence, there are natural transformations $\operatorname{Spc} \overset{\rho_S}{\Leftarrow} \operatorname{Blowup} \overset{\rho_N}{\Rightarrow} \operatorname{Nrv}$.

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then the natural maps ρ_S and ρ_N are homotopy equivalences.

Proof. Use the following up to homotopy commutative diagram

$$\begin{array}{ccccc} & & \operatorname{Blowup}(\mathcal{A}) & & \\ & \overset{\simeq}{\swarrow} & & \overset{\simeq}{\searrow} & \\ X & & \xleftarrow[\Gamma]{\simeq} & & \operatorname{Nrv}(\mathcal{A}) \end{array}$$

Unified nerve theorem

Theorem. Let X be a topological space and $\mathcal{A} = (A_i)_{i \in I}$ a cover of X.

1. Consider the natural map $\rho_S \colon \mathrm{Blowup}(\mathcal{A}) \to X$.
   a) If $\mathcal{A}$ is an open cover, then ρ_S is a weak homotopy equivalence. If furthermore X is paracompact, then ρ_S is a homotopy equivalence.
   b) Let X be compactly generated, and $\mathcal{A}$ a closed cover that is locally finite and locally finite dimensional. If for any $T \in \mathrm{Nrv}(\mathcal{A})$ the latching space $L(T) := \bigcup_{T \subsetneq J \subseteq I} A_J \subseteq A_T$ is closed and $(A_T, L(T))$ satisfies the homotopy ext. prop., then ρ_S is a homotopy equivalence.
2. Consider the natural map $\rho_N \colon \mathrm{Blowup}(\mathcal{A}) \to \mathrm{Nrv}(\mathcal{A})$.
   a) If $\mathcal{A}$ is (weakly) good, then ρ_N is a (weak) homotopy equivalence.
   b) If for all $J \in \mathrm{Nrv}(\mathcal{A})$ the space A_J is compactly generated and $\mathcal{A}$ is homologically good with respect to a coefficient ring R, then ρ_N is an R-homology isomorphism.

Unified nerve theorem

Counterexamples

1. Paracompactness is crucial to guarantee a homotopy equivalence
2. For closed good covers some “finiteness” is needed
3. The “latching assumption” is not a proof artefact; even if we only care about the homologies

Unified nerve theorem

Counterexamples

1. Paracompactness is crucial to guarantee a homotopy equivalence:

- Consider the *long ray* L
- This is a standard example for a non-paracompact space that is also not contractible
- L is weakly contractible and for any point $p \in L$ the open set $L_{<p} = \{t \in L \mid t < p\}$ is homeomorphic to the interval $[0, 1)$
- $\mathcal{A} = (L_{<p})_{p \in \omega_1}$ is a good open cover and it follows from 1.a) and 2.a) that the nerve $\mathrm{Nrv}\,\mathcal{A}$ is weakly contractible and hence contractible by Whitehead's theorem
- Thus, L and $\mathrm{Nrv}\,\mathcal{A}$ are not homotopy equivalent

Unified nerve theorem

Counterexamples

2. For closed good covers some "finiteness" is needed:

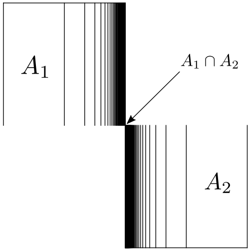
- Consider the cover of S^1 by its points
- As the nerve $\operatorname{Nrv} \mathcal{A}$ is a disjoint union of points, it is not homotopy equivalent to S^1
- All conditions in 1.b) and 2.a) are satisfied except the locally finiteness assumption

Unified nerve theorem

Counterexamples

3. The “latching assumption” is not a proof artefact:

- Consider the *double comb space* C and denote the two combs by A_1 and A_2



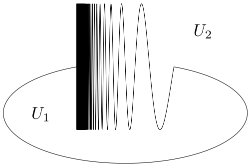
- The nerve $\operatorname{Nrv} \mathcal{A}$ is contractible, but C is not
- The pairs $(A_1, A_1 \cap A_2)$ and $(A_2, A_1 \cap A_2)$ do not satisfy the homotopy extension property

Unified nerve theorem

Counterexamples

3. The "latching assumption" is not a proof artefact; even if we only care about the homologies:

- Consider the *Warsaw circle* $W \subseteq S^2$ that separates the sphere into two connected components U_1 and U_2



- The closed sets $A_1 = U_1 \cup W$ and $A_2 = U_2 \cup W$ cover the sphere and are contractible
- $A_1 \cap A_2 = W$ is acyclic and hence $\mathcal{A} = \{A_1, A_2\}$ is a homologically good closed cover of S^2
- S^2 and $\mathrm{Nrv}\,\mathcal{A}$ do not have isomorphic homology groups

Future work

- Approximate nerve theorems
- Discuss Latschev's result on the reconstruction of Riemannian manifolds and its functoriality by using *Vietoris–Rips good covers*

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Selected references

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