Functorial nerve theorems for persistence

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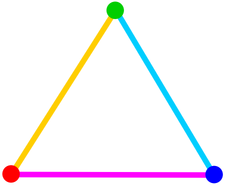
Computational Persistence 2022
November 4

joint work with Ulrich Bauer, Michael Kerber, and Alexander Rolle

The Alexandrov nerve (1928)

Definition. Let X be a topological space, and let $\mathcal{U} = (U_i)_{i \in I}$ be a cover of X. The *nerve* of $\mathcal{U}$ is the simplicial complex

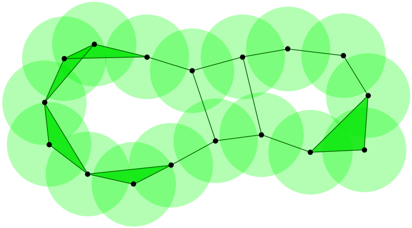
$$\mathrm{Nrv}(\mathcal{U}) = \{J \subseteq I \mid |J| < \infty \text{ and } U_J := \bigcap_{i \in J} U_i \neq \emptyset\}$$



The Alexandrov nerve (1928)

Čech complex of a point cloud

Definition. The *Čech complex* of a subset $S \subseteq \mathbb{R}^d$ is the nerve of the cover by closed balls of radius r centered at points in S

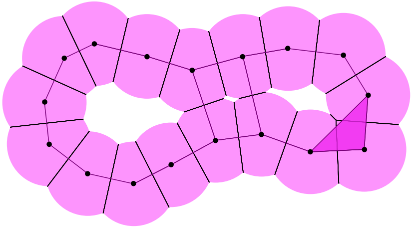


Nerve Theorem. Let $\mathcal{U}$ be an open and good cover of a paracompact space X. Then $\mathrm{Nrv}(\mathcal{U})$ is homotopy equivalent to X.

The Alexandrov nerve (1928)

Delaunay complex of a point cloud

Definition. The *Delaunay complex* of a subset $S \subseteq \mathbb{R}^d$ is the nerve of the cover by closed Voronoi balls of radius r centered at points in S



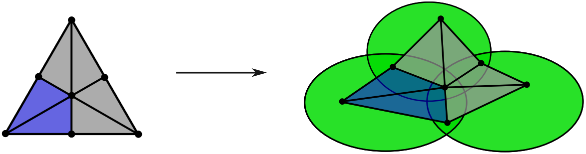
Nerve Theorem. Let $\mathcal{A}$ be a finite closed and good cover of a subspace $X \subseteq \mathbb{R}^d$. Is $\mathrm{Nrv}(\mathcal{A})$ homotopy equivalent to X?

Nerve theorem for closed convex covers

Theorem. If $X \subset \mathbb{R}^d$, and $\mathcal{A} = (C_i)_{i \in [n]}$ is a cover by closed convex subsets, then $\mathrm{Nrv}(\mathcal{A})$ is homotopy equivalent to X.

Proof strategy:

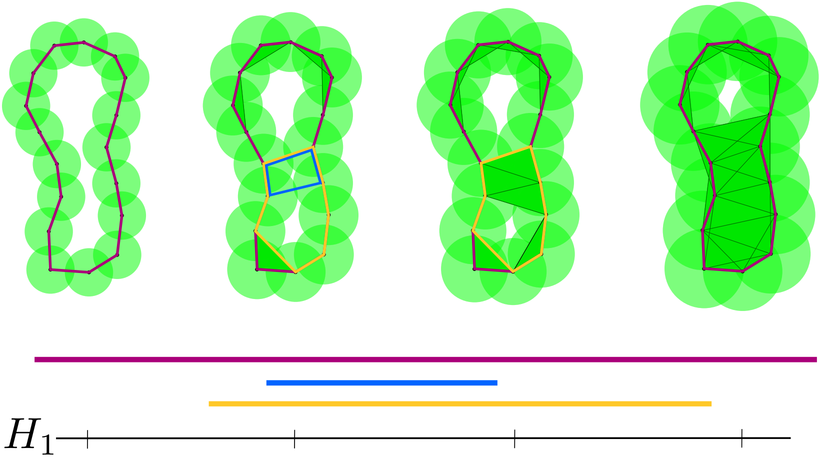
- Construct piecewise linear $\Gamma\colon \mathrm{Sd}\,\mathrm{Nrv}(\mathcal{A}) \to X$ with $\Gamma(\mathrm{bst}\, v_i) \subseteq C_i$.



- Construct $\Phi\colon X \to \mathrm{Nrv}(\mathcal{A})$ using a partition of unity subordinate to an open thickening of the C_i with $\Phi(C_i) \subseteq \mathrm{bst}\, v_i$.
- Show that Φ is a homotopy inverse to Γ:
  - $\Gamma \circ \Phi(C_i) \subseteq C_i \Rightarrow \Gamma \circ \Phi \simeq \mathrm{id}$
  - $\Phi \circ \Gamma(\mathrm{bst}\, v_i) \subseteq \mathrm{bst}\, v_i \Rightarrow \Phi \circ \Gamma \simeq \mathrm{id}$

Functoriality

Persistent homology



Functoriality

Let $X \subseteq \mathbb{R}^d$, $\mathcal{U}_r = \{B_r(x)\}_{x \in X}$, and $X_r = \bigcup \mathcal{U}_r$. The nerve theorem guarantees for $r \leq l$ the homotopy equivalences

$$\begin{array}{ccc} \mathrm{Nrv}(\mathcal{U}_r) & \longrightarrow & \mathrm{Nrv}(\mathcal{U}_l) \\ \simeq\big\uparrow & \circlearrowright? & \big\uparrow\simeq \\ X_r & \longrightarrow & X_l \end{array}$$

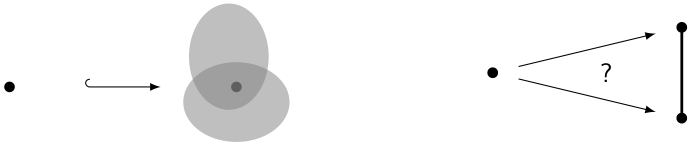
Theorem (Chazal–Oudot 2008). For open coves, this diagram commutes up to homotopy. Hence, it commutes after applying homology.

We address two issues:

1. Closed covers were not well-treated in the literature
2. No "proper" functoriality $\rightarrow$ needed in some homotopy-theoretic approaches to TDA (e.g. Blumberg–Lesnick)

Functoriality

Category of covered spaces



Definition. The *category of covered spaces* Cov has

- Obj: pairs of the form $(X, (U_i))$, with (U_i) a cover of X
- Mor: $(f, \varphi)\colon (X, (U_i)_{i \in I}) \to (Y, (V_\ell)_{\ell \in L})$, continuous map $f\colon X \to Y$ with $f(U_i) \subseteq V_{\varphi(i)}$.

Functoriality

Category of covered spaces

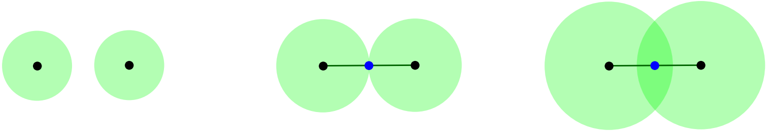
Two functors

- Forgetting the cover: $\mathrm{Spc}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto X$
- The nerve: $\mathrm{Nrv}\colon \mathsf{Cov} \to \mathsf{Top}$, $(X, \mathcal{U}) \mapsto \mathrm{Nrv}(\mathcal{U})$

Remark. There are no natural transformations between Spc and Nrv!

Functoriality

Pointed covers



Proposition. For every finite $X \subseteq \mathbb{R}^d$ there exist piecewise linear homotopy equivalences Γ_r such that for $r \leq l$ we have

$$\begin{array}{ccc} \operatorname{Sd} \check{\mathrm{C}}\mathrm{ech}_r(X) & \longrightarrow & \operatorname{Sd} \check{\mathrm{C}}\mathrm{ech}_l(X) \\ \Gamma_r \big\downarrow & \circlearrowright & \big\downarrow \Gamma_l \\ \bigcup_{x \in X} D_r(x) & \longrightarrow & \bigcup_{x \in X} D_l(x) \end{array}$$

Functoriality

Blowup complex

Definition. For a finite cover $\mathcal{U} = (U_i)_{i \in [n]}$ of X, the *blowup complex* is

$$\mathrm{Blowup}(\mathcal{U}) = \bigcup_{J \in \mathrm{Nrv}(\mathcal{U})} U_J \times \Delta^{|J|-1} \subseteq X \times \mathrm{Nrv}(\mathcal{U}) \,,$$

yielding a functor $\mathrm{Blowup}\colon$ Cov $\to$ Top.

Remark. The blowup complex is (not naturally) homeomorphic to the *bar construction* of the nerve diagram

Functoriality

Any $(f, \varphi)\colon (X, \mathcal{U}) \to (Y, \mathcal{V})$ induces a commuting diagram

$$\begin{array}{ccccc} X & \xleftarrow{\rho_S} & \mathrm{Blowup}(\mathcal{U}) & \xrightarrow{\rho_N} & \mathrm{Nrv}(\mathcal{U}) \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi_*} \\ Y & \xleftarrow[\rho_S]{} & \mathrm{Blowup}(\mathcal{V}) & \xrightarrow[\rho_N]{} & \mathrm{Nrv}(\mathcal{V}) \end{array}$$

Hence, there are natural transformations $\mathrm{Spc} \overset{\rho_S}{\Leftarrow} \mathrm{Blowup} \overset{\rho_N}{\Rightarrow} \mathrm{Nrv}$.

Unified nerve theorem

Theorem. Let X be a topological space and $\mathcal{A} = (A_i)_{i \in I}$ a cover of X.

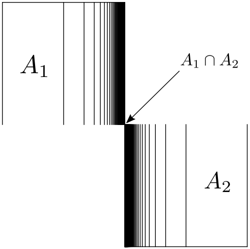
1. Consider the natural map $\rho_S \colon \operatorname{Blowup}(\mathcal{A}) \to X$.
   a) If $\mathcal{A}$ is an open cover, then ρ_S is a weak homotopy equivalence. If furthermore X is paracompact, then ρ_S is a homotopy equivalence.
   b) Let X be compactly generated, and $\mathcal{A}$ a closed cover that is locally finite and locally finite dimensional. If for any $T \in \operatorname{Nrv}(\mathcal{A})$ the latching space $L(T) := \bigcup_{T \subsetneq J \subseteq I} A_J \subseteq A_T$ is closed and $(A_T, L(T))$ satisfies the homotopy ext. prop., then ρ_S is a homotopy equivalence.
2. Consider the natural map $\rho_N \colon \operatorname{Blowup}(\mathcal{A}) \to \operatorname{Nrv}(\mathcal{A})$.
   a) If $\mathcal{A}$ is (weakly) good, then ρ_N is a (weak) homotopy equivalence.
   b) If for all $J \in \operatorname{Nrv}(\mathcal{A})$ the space A_J is compactly generated and $\mathcal{A}$ is homologically good with respect to a coefficient ring R, then ρ_N is an R-homology isomorphism.

Unified nerve theorem

Counterexamples

The “latching assumption” is not a proof artefact:

- Consider the *double comb space* C and denote the two combs by A_1 and A_2



- The nerve $\operatorname{Nrv} \mathcal{A}$ is contractible, but C is not
- The pairs $(A_1, A_1 \cap A_2)$ and $(A_2, A_1 \cap A_2)$ do not satisfy the homotopy extension property

Summary

- Sketched proof of a (functorial) nerve theorem that is attractive to students and newcomers.
- Learned that functoriality depends on the framework and holds without any additional assumptions.
- Abstract homotopy theory can help to give a “unified view” on the functorial nerve theorem.