Variations of the Nerve Theorem

Fabian Roll

Mathematics, Technical University of Munich

TopMath Mathematik mit Promotion

Technische Universität München

Abstract

The nerve theorem, whose early versions are usually attributed to Borsuk, Leray and Weil, is a classical result in algebraic topology. It states that the intersection pattern of a cover $\mathcal{A} = \{A_i\}_{i\in I}$ of a topological space X fully determines X itself in algebraic topological terms given suitable hypotheses (e.g. contractibility or acyclicity of all non-empty and finite intersections of cover elements). In this thesis, we dicuss different incarnations of the nerve theorem and see how a unified ansatz leads to their functoriality. Moreover, we were able to give a new and elementary proof in the case of compact and convex sets in $\mathbb{R}^d$, extract from a classical proof a purely chain complex nerve theorem and fill some other gaps in the literature.

Background

Since their introduction by Alexandroff [1], nerves appeared at many places in topology. With the emergence of *topological data analysis* they received increasing attention as nerves are one of the main ways to obtain a topological space, that is suitable for computations, from a point cloud. Two prominent examples are the *Čech complex* and the *Delaunay complex*, which arise as nerves of the collection of closed balls and Voronoi balls, respectively.

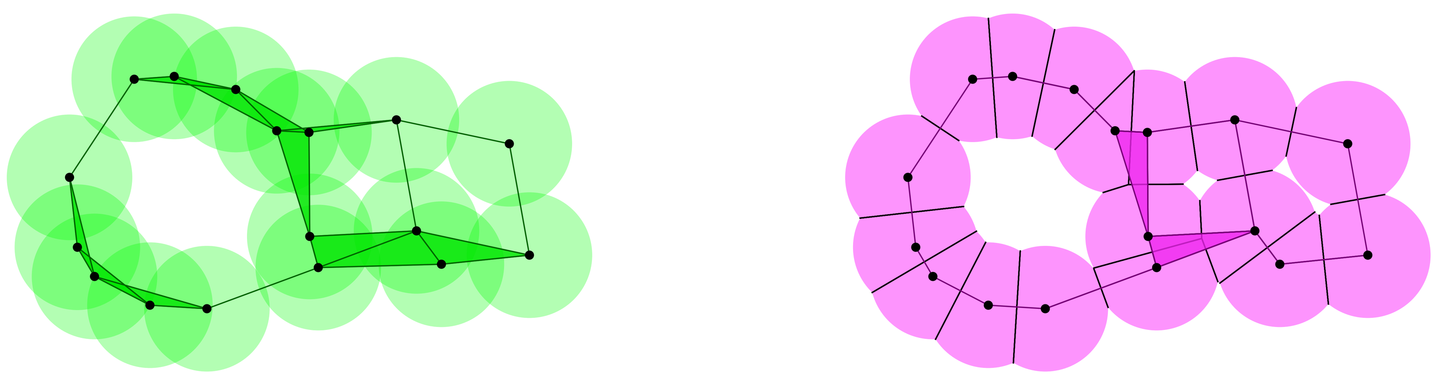

Figure 1: The Čech complex (left) and the Delaunay complex (right) for a small radius.

In this setting, the nerve theorem establishes a homotopy equivalence between the union of balls and the nerve. Moreover, the functorial nerve theorem guarantees that the inclusion of the Delaunay into the Čech complex is a homotopy equivalence and hence they have the same algebraic topological invariants (like homology). This is of high value as the Delaunay complex is much smaller than the Čech complex.

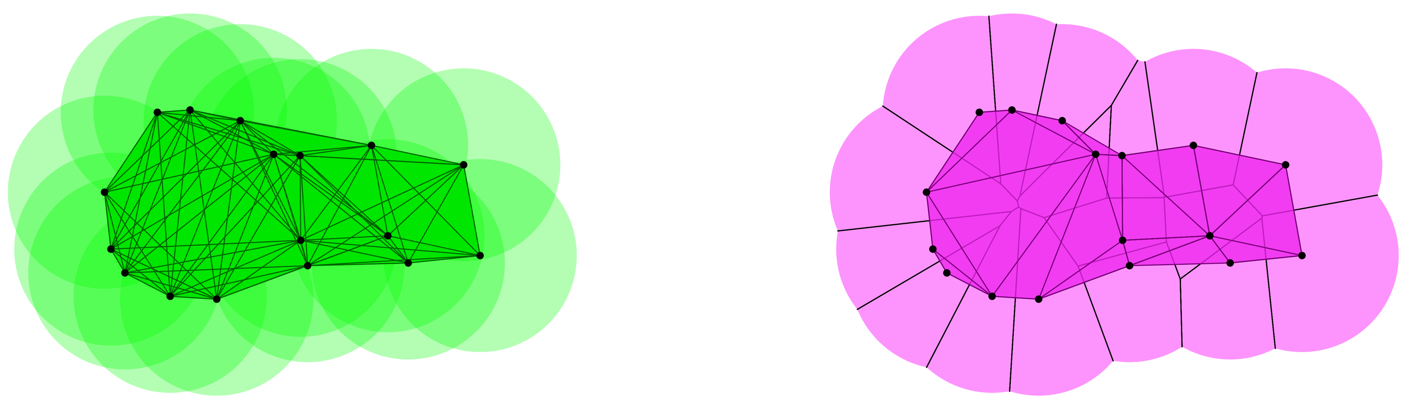

Figure 2: The Čech complex (left) and the Delaunay complex (right) for a large radius.

The literature on nerve theorems is hard to navigate. To have a better understanding of the connections between space, cover and nerve it was desirable to conceptualize the various statements and proofs. This, together with the goal to find an easy to understand proof for a nerve theorem for closed covers was the originial motivation for this thesis.

The Nerve Complex

Let $\mathcal{A} = \{A_i\}_{i\in I}$ be a family of subsets of a set X. The *nerve* of $\mathcal{A}$ is the abstract simplicial complex

$$N(\mathcal{A}) = \{J \subseteq I \mid |J| < \infty, \bigcap_{j\in J} A_j \neq \emptyset\}.$$

One obtains a topological space — more precisely, a simplicial complex — $|N(\mathcal{A})|$ by geometric realization (space built according to the combinatorial data) [4].

Example. Consider the cover $\{r, g, b\}$ of the circle by three open arcs. Then, the nerve is given by $\{\{r\}, \{g\}, \{b\}, \{r, g\}, \{r, b\}, \{g, b\}\}$.

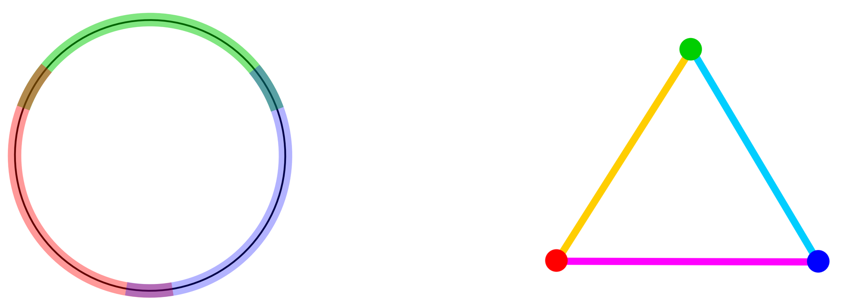

Figure 3: The cover (left) and the associated geometric realization of its nerve (right).

Unified Approach to Nerve Theorems

Let X be a topological space and $\mathcal{A} = \{A_i\}_{i\in I}$ a cover. Then, one can construct a zigzag

$$X \xleftarrow{p_f} \operatorname{hocolim} \mathcal{D}_{\mathcal{A}} \xrightarrow{p_b} |N(\mathcal{A})|, \tag{i}$$

where $\operatorname{hocolim} \mathcal{D}_{\mathcal{A}}$ is the *homotopy colimit* of the *nerve diagram* [4,5].

Main Takeaway. To show that X and $|N(\mathcal{A})|$ are (weakly) homotopy equivalent or have isomorphic homology it is sufficient to prove that p_f and p_b are (weak) homotopy equivalences or homology isomorphisms, respectively.

A Zoo of Nerve Theorems

The unified approach allows us to study p_f and p_b in Eq. (i) independently of each other.

Theorem. If $\mathcal{A}$ is a *good cover*, i.e. every non-empty and finite intersection of cover elements is (weakly) homotopy equivalent or homology isomorphic to the one point space, then p_b is a (weak) homotopy equivalence or homology isomorphism, respectively.

Theorem. The map p_f is a weak homotopy equivalence if $\mathcal{A}$ is an open cover and a homotopy equivalence if either of the following holds

- X is paracompact and $\mathcal{A}$ is an open cover
- X is simplicial complex and $\mathcal{A}$ a cover by subcomplexes
- X is a subset of $\mathbb{R}^d$ and $\mathcal{A}$ is a finite cover by compact and convex subsets

Applications.

- Under the assumptions above, the (co)homology of X is isomorphic to the (co)homology of the nerve of the cover.
- Every smooth manifold admits a geodesically convex cover. Thus, every smooth manifold is homotopy equivalent to the geometric realization of a simplicial complex.

Functoriality of Nerve Theorems

To analyze nerves and their functorial behaviour it is convenient to consider *maps between covered spaces*. Such a morphism $(f, C) : (X, \mathcal{A} = \{A_i\}_{i\in I}) \to (Y, \mathcal{B} = \{B_j\}_{j\in J})$ consists of a continuous map $f : X \to Y$ and a set map $C : I \to J$ such that for any $i \in I$ we have

$$f(A_i) \subseteq B_{C(i)}.$$

A map between covered spaces induces a commutative diagram

$$\begin{tikzcd} X \arrow[d] & \operatorname{hocolim} \mathcal{D}_{\mathcal{A}} \arrow[l] \arrow[r] \arrow[d] & {|N(\mathcal{A})|} \arrow[d] \\ Y & \operatorname{hocolim} \mathcal{D}_{\mathcal{B}} \arrow[l] \arrow[r] & {|N(\mathcal{B})|.} \end{tikzcd} \tag{ii}$$

Applications.

- For a good cover filtration the persistent (singular) homology is isomorphic to the persistent (simplicial) homology of the induced filtration of the nerves.
- It is a direct consequence of the functorial nerve theorem that singular (co)homology coincides with Čech (co)homology (coefficients in a fixed abelian group) on smooth manifolds.

Future Work

- The unified approach leads to a *unified nerve theorem*. In particular, one can establish a nerve theorem for closed covers by using model categories. [6]
- It is a general scheme in topological data analysis to adapt classical notions of (algebraic) topology to an approximate setting. Hence, it is tempting to relax the notion of a good cover to an ϵ-good cover. We plan to analyze known *approximate nerve theorems* and their proofs to find a conceptual explanation for these results.

Selected References

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